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Coherence and Incompatability in W^*-Algebraic Quantum Theory

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In the framework of generalized quantum theory using a W^*-algebraic formalism, we introduce a completely symmetric coherence relation for states which is also applicable to nonpure states. Making use of lattice theoretic results the properties of this relation, especially its connection with incompatibility, are investigated. By means of algebraic decomposition theory the investigation is reduced to the case of factors where a complete classification of the coherence classes is given.

1. INTRODUCTION

There are mainly two features which exemplify the drastic change in conceptions quantum theory constitutes with respect to classical physics: The occurrence of principally incompatible observables and the possibility of coherent state superpositions. It is well known how to formulate mathematically these features in traditional Hilbert space quantum mechanics. But also in more general descriptions it is important to clarify all questions connected with these basic structures.

Here and in the following let us understand by a "description" the mathematical specification of the set of observables and the set of states in a physical theory, together with the duality relation, which gives the (expectation) values of the observables in the states. To every description belongs the group of structural symmetries consisting of those transformations in one of these sets which can be compensated for by a dual (and inverted) transformation in the other set. The simultaneous application of both

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transformations leaves then the physical content of the theory unchanged. In this terminology a closed dynamical system would consist of a description in which a one-parameter group of structural symmetries is singled out as the set of time translations.

The term "traditional quantum mechanics" will here be used for the description which has $\sum_n^{\oplus}\mathfrak{B}(\mathfrak{H}_n)$, where $\mathfrak{B}(\mathfrak{H}_n)$ stands for the set of all bounded linear operators on the (separable) Hilbert space $\mathfrak{H}_n$, as the set of observables and in which the states are given by the set of all density operators in $\sum_n^{\oplus}\mathfrak{H}_n$. The $\mathfrak{H}_n$ are usually called coherent subspaces. For foundational investigations and for the study of systems with infinitely many degrees of freedom or systems having classical and quantum features more general descriptions have been developed. If one concentrates on yes–no experiments one comes along with an orthocomplemented (orthomodular) lattice for the set of observables. Such a lattice description, where the states are some set of positive normalized, completely orthoadditive functions on the lattice, is called a quantum logic (Jauch, 1968; Varadarajan, 1968; Piron, 1976). More operationalistic approaches require a description where the observables are at least embedded in the dual of an ordered vector space (Ludwig, 1970; Davies and Lewis, 1970; Edwards, 1970). The various algebraic descriptions arise by abstraction and generalization of the operator algebras in traditional quantum mechanics to W^* and C^* (among other) algebras (Emch, 1972).

We employ throughout the paper the W^*-algebraic description, where the observables are given as (self-adjoint) elements of a (Hilbert-space-independent) W^* algebra and the states by all positive, normalized, normal linear functionals of the algebra. This description is distinguished by its formal elegance and, in connection with this, by having the richest structure of all space-independent descriptions. For a more fundamental motivational discussion of W^*-algebraic descriptions we refer to (Primas, 1980, 1981). Moreover, many investigations in a C^*-algebraic description can be mapped into a W^*-algebraic framework. Complicated physical systems such as macromolecules and many-body systems require descriptions with arbitrarily complex W^* algebras, the centers of which represent the classical observables. The limiting case of purely classical systems is described by an Abelian W^* algebra. Thus we carry out our investigation in a general W^* algebra, the only restrictive assumption (that of a separable predual) coming into play when applying the central decomposition theory.

In the exposition of the required mathematical formalism for W^*-algebraic descriptions (Section 2) emphasis is laid on the properties of supports, lattice theoretical notions, and decomposition theory.

Starting from the special case of pure states in Section 3 a Hilbert-space-independent formulation of a coherence relation for three arbitrary (normal) states is proposed. This generality is required by the facts that, firstly, the observable algebras of complicated physical systems have many not unitarily equivalent representations—at least in a certain approximation —so that the fundamental structures should belong to the algebraic part of the description and that, secondly, the pure states play a much less important role then, than in traditional quantum theory. The possible presence of many classical observables leads also to state relations, where quantum mechanical coherent superpositions are classically mixed. But at first we carry out only the formal investigation of the coherence relation, make explicit its connection with lattice theoretic notions, and show that it is intimately related to incompatibility. Further on, the coherence relation is decomposed into relations for factors and an appropriate generalization of a coherent sector is worked out.

In Section 4 we specialize to factors and classify there the coherence classes.

In the conclusions we give a first physical interpretation of the obtained results.

As for the literature concerning the coherent superposition of states one must mention the classical exposition (Dirac, 1930) where the unrestricted superposition principle is used to find the appropriate mathematical state concept for microsystems. It seems that the first systematic discussion of a quantum system with superselection rules, where Dirac's superposition principle is thus broken, was performed as late as in Wick et al. (1952). Jauch (Jauch and Misra, 1964; Jauch, 1968) not only elaborated the general theory of quantum descriptions with superselection rules but also put forward a completely new formulation of the coherent superposition relation referring it to atomic propositions of quantum logics. Also in quantum logics but now referring to states, Varadarajan (1968) formulated a notion of superposition and a version of a superposition principle which both are not characteristic for quantum mechanics. Gudder (1970), and Pulmannova (1980) also deal with Varadarajan's formulation. For pure states of a C^*-algebraic description the coherent superposition relation is examined in Roberts and Roepstroff (1969). The only reference we know of, in which it is also proposed to apply Jauch's superposition relation to the support projections of normal states in W^*-algebraic descriptions is Chen (1973). This paper contains some erroneous statements, notably Proposition 1.

2. FORMALISM OF W^*-ALGEBRAIC DESCRIPTIONS

For later application we collect here some mathematical formalism of a W^*-algebraic description, that is a description which is based on a W^* algebra $\mathfrak{M}$ and the set $\mathfrak{S}_n$ of all normal states of $\mathfrak{M}$. Here and in the following, the word state used in a mathematical context will always mean normal, positive, normalized linear functional. For general information on W^*-algebras, we refer to the standard textbooks (Dixmier, 1969; Sakai, 1971; Strătilă and Zsidó, 1979; and Takesaki, 1979).

The group $\mathfrak{G}$ of structural symmetries of this description consists of all affine bijections ν: $\mathfrak{S}_n \to \mathfrak{S}_n$. By a slight modification of the reasoning in Kadison (1965) one concludes that for $\nu \in \mathfrak{G}$ the dual mapping $\alpha = \nu^*$: $\mathfrak{M} \to \mathfrak{M}$ is a Jordan-$*$-automorphism.

The set of all projections (i.e., self-adjoint idempotents) in $\mathfrak{M}$ is denoted by $P(\mathfrak{M})$, and the lattice operations may be introduced algebraically as follows $[P, Q \in P(\mathfrak{M})]$:

$$P \leqslant Q \qquad \text{if } PQ = P \tag{1}$$

$$P^{\perp} = 1 - P, \qquad \text{where 1 is the multiplicative identity in } \mathfrak{M}. \tag{2}$$

$$P \wedge Q = \sigma(\mathfrak{M}, \mathfrak{M}_*) - \lim_{n \to \infty} (PQP)^n = \inf_{n \in \mathrm{N}} (PQP)^n \tag{3}$$

$$P \vee Q = 1 - (P^{\perp} \wedge Q^{\perp}) \tag{4}$$

With these definitions $P(\mathfrak{M})$ is a complete lattice with orthocomplementation $\perp$. A Jordan-$*$-automorphism restricted to $P(\mathfrak{M})$ constitutes an ortholattice automorphism, that is a bijection of $P(\mathfrak{M})$ which preserves the operations $\perp$ and $\wedge$ (and thus also $\leqslant$ and $\vee$). Conversely, every ortholattice automorphism of $P(\mathfrak{M})$ may be extended to a Jordan-$*$-automorphism of $\mathfrak{M}$ provided $\mathfrak{M}$ has no direct summand of type I_2 (Dye, 1955). A Jordan-$*$-automorphism is automatically $\sigma(\mathfrak{M}, \mathfrak{M}_*)$-continuous, and its dual maps the normal states of $\mathfrak{M}$ one-to-one and onto themselves, and is thus a structural symmetry.

Associated with each state $\rho \in \mathfrak{S}_n$ there are two very useful projections, the support of ρ

$$S_\rho = \inf\{P \in P(\mathfrak{M}) : \rho(PAP) = \rho(A), \forall A \in \mathfrak{M}\} \tag{5}$$

and the central support of ρ

$$C_\rho = \inf\{P \in P(\mathfrak{Z}) : \rho(PAP) = \rho(A), \forall A \in \mathfrak{M}\} \tag{6}$$

where $\mathfrak{Z}$ denotes the center of $\mathfrak{M}$. By normality, $\rho(S_\rho) = \rho(C_\rho) = 1$, and these are the smallest projections in $P(\mathfrak{M})$ and $P(\mathfrak{Z})$, respectively, with this property. Furthermore, $C_\rho = \inf\{P \in P(\mathfrak{Z}) : S_\rho \leqslant P\}$ is the central cover of S_ρ. ρ is pure iff S_ρ is minimal in $P(\mathfrak{M})$, and only in this case is ρ uniquely determined by S_ρ. ρ is factorial (i.e., the corresponding GNS representation is a factor) iff C_ρ is minimal in $P(\mathfrak{Z})$. Two states ρ and φ are quasiequivalent iff $C_\rho = C_\varphi$. ρ is faithful, iff $S_\rho = 1$. S_ρ and C_ρ are always σ-finite (or countably decomposable), that is they dominate an at most countable family of pairwise orthogonal nonzero projections. Conversely, if $P \in P(\mathfrak{M})$ is σ-finite, then there is a (normal) state ρ, such that $S_\rho = P$.

Physically, the support S_ρ may be considered as the smallest filter which ρ passes unchanged, and C_ρ as the smallest of these filters which can be devised by means of classical observables only.

For $\nu \in \mathfrak{G}$, and with $\alpha = \nu^*$, one has

$$S_{\nu(\rho)} = \alpha^{-1}(S_\rho) \tag{7}$$

$$C_{\nu(\rho)} = \alpha^{-1}(C_\rho) \tag{8}$$

for every $\rho \in \mathfrak{S}_n$. If π: $\mathfrak{M} \to \mathfrak{M}_\pi \subseteq \mathfrak{B}(\mathfrak{H}_\pi)$ is a W^* isomorphism of $\mathfrak{M}$ onto a von Neumann algebra over a Hilbert space $\mathfrak{H}_\pi$, then π^* maps $\mathfrak{S}_n(\mathfrak{M}_\pi)$ affinely and bijectively onto $\mathfrak{S}_n(\mathfrak{M})$, and we have for the support of $\rho_\pi = (\pi^*)^{-1}(\rho) = \rho \circ \pi^{-1}$

$$S_{\rho_\pi} = \pi(S_\rho) \tag{9}$$

for every $\rho \in \mathfrak{S}_n(\mathfrak{M})$. Let the density operator D_π of ρ_π (which need not be unique) have the spectral decomposition $D_\pi = \Sigma_n \lambda_n P_n$, $\lambda_n > 0$; then

$$\begin{aligned} S_{\rho_\pi} &= \inf\{P \in P(\mathfrak{M}_\pi) : P \geq \vee_n P_n\} \\ &= [\mathfrak{M}'_\pi(\vee_n P_n)\mathfrak{H}_\pi] \end{aligned} \tag{10}$$

where [...] stands for the projection onto the smallest closed subspace containing the set in the bracket and a prime denotes the commutant in $\mathfrak{B}(\mathfrak{H}_\pi)$ (Dixmier, 1969, p. 5).

In order to understand the peculiarities of $P(\mathfrak{M})$ it is useful to cite first some notions of the theory of general orthomodular lattices. Let $\mathfrak{L}$ be a lattice under the ordering $\leq$ with the universal bounds 0 and 1. A map $^\perp$: $\mathfrak{L} \to \mathfrak{L}$ is called an orthocomplementation if

(i) $P \leq Q$ implies $Q^\perp \leq P^\perp$ (11)

(ii) $(P^\perp)^\perp = P, \qquad \forall P \in \mathfrak{L}$ (12)

(iii) $P \wedge P^\perp = 0, \qquad \forall P \in \mathfrak{L}$ (13)

Then one concludes that $0^{\perp}=1$, $\perp$ is one-to-one, and

$$(P \vee Q)^{\perp}=P^{\perp} \wedge Q^{\perp} \tag{14}$$

One easily verifies that (2) fulfills (11)–(13).

Let us call an ordered triple (P, Q, R) of elements of $\mathfrak{L}$ distributive if

$$P \wedge (Q \vee R)=(P \wedge Q) \vee (P \wedge R) \tag{15}$$

$\mathfrak{L}$ is called distributive if all its triples are distributive. In orthocomplemented lattices one has the compatibility relation (Birkhoff and von Neumann, 1936)

$$P\kappa Q, \text{ if } (P, Q, Q^{\perp}) \text{ is distributive} \tag{16}$$

This relation is symmetric iff the orthocomplemented lattice satisfies a very weak distributivity property, the so-called orthomodularity condition:

$$Q \leqslant P \text{ implies } P\kappa Q \tag{17}$$

(Nakamura, 1957), a condition which is satisfied in $P(\mathfrak{M})$. In an orthomodular lattice [i.e., an orthocomplemented lattice satisfying (17)] one has furthermore (Piron, 1964)

$$P\kappa Q, \text{ iff } C(P, Q) \equiv (P \wedge Q) \vee (P \wedge Q^{\perp}) \vee (P^{\perp} \wedge Q) \vee (P^{\perp} \wedge Q^{\perp}) = 1 \tag{18}$$

and from this

$$P\kappa Q, \quad \text{iff the smallest sublattice which contains } P \text{ and } Q \text{ is distributive} \tag{19}$$

In $P(\mathfrak{M})$ we have a simple characterization of compatibility

$$P\kappa Q, \qquad \text{iff } [P, Q]_{-}=0 \tag{20}$$

Thus $P(\mathfrak{M})$ is distributive iff $\mathfrak{M}$ is Abelian. The lattice expression $C(P, Q)$ associates to a pair P and Q, an element in $P(\mathfrak{M})$ ranging from 0 to 1. From (18) one is led to interpret the smallness of $C(P, Q)$ as a measure of the incompatibility of P and Q. Thus we propose the following definition.

2.1. Definition. Two elements P and Q in an orthocomplemented lattice are called maximally incompatible if $C(P, Q)=0$.

We see that $C(P,Q)=0$ iff all four intersections $P^{(\perp)}\wedge Q^{(\perp)}=0$, which was denoted P and Q are in "position P" in Dixmier (1948).

In the (quantum) logical interpretation of a (nondistributive) orthomodular lattice the elements of the lattice are called "propositions." A proposition P is said to be "truth definite" or "objective" in a state if it has the expectation value 0 or 1; in the latter case it is also said to be "actualized." In a pure state of a nondistributive lattice the one of two compatible propositions is truth definite if the other is so. In general the propositions have expectation values in the interval [0, 1] and the logical interpretation of the lattice operations requires some caution. In an orthomodular lattice $P\leqslant Q$ and $P\leqslant Q^{\perp}$ imply each $P\kappa Q$, so that $\leqslant$ and $\perp$ can indeed be interpreted as "implication" and "negation," respectively.

In pursuing the point further we specialize to $\mathfrak{L}=P(\mathfrak{M})$ and consider a state $\rho\in\mathfrak{S}_n$. The set

$$\mathfrak{T}_\rho=\{P\in P(\mathfrak{M}):\rho(P)\in\{0,1\}\} \tag{21}$$

is then a complete orthomodular sublattice of $P(\mathfrak{M})$, and the restriction of ρ to $\mathfrak{T}_\rho$ is an ortholattice homomorphism of $\mathfrak{T}_\rho$ onto the Boolean algebra $\{0,1\}=P(\mathfrak{M}=\mathbb{C})$. With this background information the meaning of maximal incompatibility is clarified as follows.

2.2. Proposition. Two propositions $P,Q\in P(\mathfrak{M})$ are maximally incompatible, iff they cannot be simultaneously truth definite in any state.

Proof. Suppose $\{P,P^{\perp},Q,Q^{\perp}\}\subseteq\mathfrak{T}_\rho$ for some state ρ. Then there is a pair R, $S\in\{P,P^{\perp},Q,Q^{\perp}\}$ with $\rho(R)=\rho(S)=1$, and then $\rho(R\wedge S)=1$, which gives a contradiction if $C(P,Q)=0$. Thus $C(P,Q)>0$. If $C(P,Q)>0$, then for some pair R, $S\in\{P,P^{\perp},Q,Q^{\perp}\}$, $R\wedge S>0$, and there is a state ρ with $\rho(R\wedge S)=1$ for $R\wedge S$ dominates at least one σ-finite projection. Then $\rho(R)=\rho(S)=1$ because both projections dominate $R\wedge S$, and $\rho(R^{\perp})=\rho(S^{\perp})=0$, which implies $\{P,P^{\perp},Q,Q^{\perp}\}\subseteq\mathfrak{T}_\rho$. ∎

The quantum logical investigations above did not make much use of the fact that $P(\mathfrak{M})$ is obtained from a W^* algebra. This, however, comes into play in the equivalence relation of von Neumann and Murray, defined as: $P\sim Q$, if there is a $W\in\mathfrak{M}$ with $P=W^*W$ and $Q=WW^*$, which automatically implies that W can be chosen to be a partial isometry. If there is a unitary element $U\in\mathfrak{M}$, such that $Q=UPU^*$, we say that P and Q are unitarily equivalent and write $P\overset{u}{\sim}Q$. $P\overset{u}{\sim}Q$ holds iff $P\sim Q$ and $P^{\perp}\sim Q^{\perp}$ hold. Let us recall some facts concerning the equivalence relation $\sim$ in $P(\mathfrak{M})$, especially the parallelogram law

$$(P\vee Q)\wedge P^{\perp}=(P\vee Q)-P\sim Q-(P\wedge Q)=Q\wedge(P^{\perp}\vee Q^{\perp}) \tag{22}$$

valid for all pairs $P, Q \in P(\mathfrak{M})$. One writes $P \prec Q$ if there is a projection $R \sim P$ with $R \leqslant Q$, and finds $P \prec Q$ and $Q \prec P$ implies $P \sim Q$. If $\mathfrak{M}$ is a factor (i.e., has a trivial center) the $\sim$-equivalence classes are totally ordered by the natural order induced by $\prec$. This total order is mapped order-isomorphically by the dimension function D into the extended nonnegative reals, the range of this mapping being characteristic for the type of the factor. Viewed as a function on $P(\mathfrak{M})$ the dimension function is characterized uniquely (up to a positive dilation) by the properties

$$D(P) = 0, \qquad \text{iff } P = 0 \tag{23}$$

$$P \perp Q \text{ (i.e., } P \leqslant Q^{\perp}\text{) implies } D(P+Q) = D(P) + D(Q) \tag{24}$$

$$P \text{ is finite (i.e., } P \sim Q, Q \leqslant P \text{ imply } P = Q\text{) iff } D(P) < \infty \tag{25}$$

$$P \sim Q, \qquad \text{iff } D(P) = D(Q) \tag{26}$$

Using (22) we obtain

$$D(P) + D(Q) = D(P \wedge Q) + D(P \vee Q) \tag{27}$$

As an attempt to introduce a purely lattice theoretic version of an equivalence relation von Neumann (1960) introduced the notion of perspectivity, a modification of which is strong perspectivity.

2.3. Definition. Two elements P, Q in an orthocomplemented lattice are called perspective (via a third element R) if

$$P \wedge R = Q \wedge R = 0 \quad \text{and} \quad P \vee R = Q \vee R = 1 \tag{28}$$

for some R in the lattice. P and Q are called strongly perspective (via R) if

$$P \wedge R = Q \wedge R = 0 \quad \text{and} \quad P \vee R = Q \vee R = P \vee Q \tag{29}$$

for some R in the lattice.

If an orthomodular lattice P and Q are strongly perspective via R then they are perspective via $R \vee (P^{\perp} \wedge Q^{\perp})$ (whereas perspectivity implies strong perspectivity only in a so-called modular lattice). The decisive property of perspectivity to use it for the introduction of a dimension function and for classificatorial properties would be its transitivity, which is, however, not valid in a general orthomodular lattice (Holland, 1970). In the special orthomodular lattice $P(\mathfrak{M})$ of an arbitrary W^* algebra $\mathfrak{M}$ one has the following:

> *2.4. Lemma* (Fillmore, 1965). If for $P, Q \in P(\mathfrak{M})$, $P \wedge Q = 0$ and $P \sim Q$, then P and Q are strongly perspective.

With this one can show the following:

> 2.5. *Theorem* (Fillmore, 1965). Two elements in $P(\mathfrak{M})$ are perspective iff they are unitarily equivalent.

These two results, which will be of fundamental importance for our study of the coherence relation, reveal indeed an intimate relationship between perspectivity and equivalence and show the transitivity of the former relation. Let us mention a simple but useful consequence thereof.

> 2.6. *Corollary*. For two elements $P, Q \in P(\mathfrak{M})$ equivalence $(P \sim Q)$ and trivial intersection $(P \wedge Q = 0)$ imply unitary equivalence $(P \overset{u}{\sim} Q)$.

In order to apply the algebraic decomposition theory also to the lattice operations and relations, we have to assume that $\mathfrak{M}$ has a separable predual (i.e., can be faithfully represented as a von Neumann algebra over a *separable* Hilbert space), and we use the following setup: We start from a faithful (normal) state φ of $\mathfrak{M}$ and construct the central measure μ of φ (For an exposition of algebraic decomposition theory, cf. Bratteli and Robinson, 1979; and Takesaki, 1979). Since $\mathfrak{S}_n$ is separable μ is supported (and not only quasisupported) by factor states. $\mathfrak{M}$ is then W^*-isomorphic to $\mathfrak{M}_\varphi = \pi_\varphi(\mathfrak{M})$, where $(\pi_\varphi, \mathfrak{H}_\varphi, \Omega_\varphi)$ is the GNS triplet corresponding to φ. The GNS triplet is unitary equivalent, and thus will be identified with a triplet of decomposable objects. We write

$$\mathfrak{H}_\varphi = \int^\oplus \mathfrak{H}_\omega \, d\mu(\omega) \tag{30}$$

$$\Omega_\varphi = \int^\oplus \Omega_\omega \, d\mu(\omega) \tag{31}$$

$$\mathfrak{M}_\varphi = \int^\oplus \mathfrak{M}_\omega \, d\mu(\omega) \tag{32}$$

where again $\mathfrak{M}_\omega = \pi_\omega(\mathfrak{M})$ and $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ is the GNS triplet for arbitrary $\omega \in \mathfrak{S}_n$. An operator $A \in \mathfrak{B}(\mathfrak{H}_\varphi)$ is in $\mathfrak{M}_\varphi$, iff there exists a μ-measurable, essentially bounded field of operators $\{A^\omega\colon A^\omega \in \mathfrak{B}(\mathfrak{H}_\omega), \omega \in \mathfrak{S}_n\}$ such that for $\Psi = \int^\oplus \Psi_\omega \, d\mu(\omega) \in \mathfrak{H}_\varphi$, $A\Psi = \int^\oplus A^\omega \Psi_\omega \, d\mu(\omega)$. We then have the pointwise decomposition of all $*$-algebraic operations in $\mathfrak{M}_\varphi$, and in virtue of (1)–(4) the lattice operations are pointwise decomposed. Since every $\rho \in \mathfrak{S}_n$ may be viewed as an element in $\mathfrak{S}_n(\mathfrak{M}_\varphi)$, it is associated with a (μ-a.e. unique) integrable field $\{\rho^\omega \in \mathfrak{S}_n(\mathfrak{M}_\omega)\colon \omega \in \mathfrak{S}_n\}$ such that for all $A \in \mathfrak{M}$, we have

$$\rho(A) = \int \rho^{\omega}(A^{\omega})\, d\mu(\omega) \tag{33}$$

We then write

$$\rho = \int^{\oplus} \rho^{\omega}\, d\mu(\omega) \tag{34}$$

The following results are either directly from von Neumann (1949), or are readily derived from this fundamental paper.

2.7. Proposition. Let $\mathfrak{M}$ be a W^* algebra with a separable predual. Identifying $\mathfrak{M}$ with the central decomposition (32) of $\mathfrak{M}_{\varphi}$ acting on (30), where φ is any faithful (normal) state of $\mathfrak{M}$, one has

(i) $\int^{\oplus} P^{\omega}\, d\mu(\omega) = P \in P(\mathfrak{M})$, iff $P^{\omega} \in P(\mathfrak{M}_{\omega})$ μ-a.e. $(P^{\perp})^{\omega} = (P^{\omega})^{\perp} = 1^{\omega} - P^{\omega}$ μ-a.e.

(ii) If $P, Q \in P(\mathfrak{M})$, $P = \int^{\oplus} P^{\omega}\, d\mu(\omega)$ and $Q = \int^{\oplus} Q^{\omega}\, d\mu(\omega)$, then

$$P \leq Q, \text{ iff } P^{\omega} \leq Q^{\omega}\ \mu\text{-a.e.} \tag{35}$$

$$P \wedge Q = \int^{\oplus} (P^{\omega} \wedge Q^{\omega})\, d\mu(\omega) \tag{36}$$

$$P \prec Q, \text{ iff } P^{\omega} \prec Q^{\omega}\ \mu\text{-a.e.} \tag{37}$$

$$P \overset{\mu}{\sim} Q, \text{ iff } P^{\omega} \overset{\mu}{\sim} Q^{\omega}\ \mu\text{-a.e.} \tag{38}$$

(iii) If $\int^{\oplus} \rho^{\omega}\, d\mu(\omega) = \rho \in \mathfrak{S}_n$, then $(S_{\rho})^{\omega} = S_{\rho^{\omega}}$, and $(C_{\rho}^{\omega}) = C_{\rho^{\omega}}$ μ-a.e.

(iv) If $\int^{\oplus} P^{\omega}\, d\mu(\omega) \in P(\mathfrak{M})$, then $D(P^{\omega})$ is a μ-measurable function.

3. THE GENERAL COHERENCE RELATION

Let us begin our discussion of coherent superposition in an abstract W^*-algebraic description by adapting to this formalism the definition of superposition given originally in Varadarajan (1968) within a quantum logical context (and then repeatedly rediscussed in the literature): Given two possibly mixed states ρ_1 and ρ_2 of a W^* algebra $\mathfrak{M}$, we say that a third state ρ_3 is a superposition of ρ_1 and ρ_2, if for $P \in P(\mathfrak{M})$

$$\rho_i(P) = 0, \qquad i = 1,2, \text{ implies } \rho_3(P) = 0 \tag{39}$$

This relation is easily seen to be satisfied if $\rho_3 = \lambda\rho_1 + (1-\lambda)\rho_2$, $\lambda \in [0,1]$, i.e., if ρ_3 is a mixture of ρ_1 and ρ_2. It can be shown that for commutative $\mathfrak{M}$ this is indeed the only possible form of superposition, and if one requires additionally all involved states to be pure then necessarily $\lambda = 0$ or 1 (Varadarajan, 1968).

Let S_i be the support of ρ_i and recall that $S_i^{\perp}$ is the largest projection with $\rho_i(S_i^{\perp}) = 0$. Then (39) can be written $P \leqslant S_i^{\perp}$, $i = 1,2$, implies $P \leqslant S_3^{\perp}$; or $S_1^{\perp} \wedge S_2^{\perp} \leqslant S_3^{\perp}$; or, finally

$$S_3 \leqslant S_1 \vee S_2 \tag{40}$$

In traditional quantum mechanics, where $\mathfrak{M} = \Sigma_n^{\oplus}\mathfrak{B}(\mathfrak{H}_n)$, three states ρ_i are pure iff they are given by state vectors Ψ_i, $i = 1,2,3$, of $\Sigma_n^{\oplus}\mathfrak{H}_n$ which lie, respectively, in one coherent subspace. Since then S_i is the projection onto the one-dimensional subspace spanned by Ψ_i, (40) is equivalent to

$$\Psi_3 = c_1\Psi_1 + c_2\Psi_2, \qquad c_i \in \mathbb{C} \tag{41}$$

and coincides with the usual Hilbert space notion of coherent superposition of vector states in a coherent subspace $\mathfrak{H}_n$. Before entering the general discussion note that for pure states (39) is only nontrivial if the states are distinct or, equivalently, have nonintersecting supports.

3.1. Proposition. Let ρ_i, $i = 1,2,3$, be pure states of a W^* algebra. Then ρ_3 is a nontrivial superposition of ρ_1 and ρ_2, iff

$$S_1 \wedge S_2 = S_1 \wedge S_3 = S_2 \wedge S_3 = 0 \quad \text{and} \quad S_1 \vee S_2 = S_1 \vee S_3 = S_2 \vee S_3 \tag{42}$$

where S_i is the support of ρ_i.

Proof. Assuming (42) one has $S_1 \vee S_2 = S_1 \vee S_3 \geqslant S_3$ and thus (40). Conversely, assume (40) and nontriviality ($\rho_1 \neq \rho_2$, $\rho_1 \neq \rho_3$ and $\rho_2 \neq \rho_3$). Then since the S_i are minimal in $P(\mathfrak{M})$ and characterize the ρ_i uniquely, we have $S_i \wedge S_j = 0$ for $1 \leqslant i \neq j \leqslant 3$, giving the first part of (42). Define $Q_i = S_i \vee S_3$, $i = 1,2$. Then $S_3 \leqslant Q_i \leqslant S_1 \vee S_2$ by (40), and $S_1 \vee Q_2 = S_2 \vee Q_1 = Q_1 \vee Q_2 = S_1 \vee S_2$. So to establish the second part of (42) we have to show that $S_1 \leqslant Q_2$ and that $S_2 \leqslant Q_1$. By atomicity, we have $S_i \wedge Q_j = S_i$ or 0, for $1 \leqslant i \neq j \leqslant 2$. Suppose that $S_1 \wedge Q_2 = 0$, then using (22): $Q_2 = Q_2 - (S_1 \wedge Q_2) \sim (S_1 \vee Q_2) - S_1 = (S_1 \vee S_2) - S_1 \sim S_2 - (S_1 \wedge S_2) = S_2$. Thus Q_2 is an atom, which contradicts $0 < S_2 \leqslant Q_2$ unless $S_2 \leqslant S_3$, which is not the case.

This shows that $S_1 \wedge Q_2 = S_1$ or $S_1 \leqslant Q_2$. Repeating the same argument for $S_2 \wedge Q_1$, one obtains $S_2 \leqslant Q_1$. ■

Proposition 3.1 reveals the fact that not only in traditional quantum mechanics but quite generally the apparently asymmetric superposition relation (39) is for pure states equivalent to the completely symmetric relation (42), or trivial. This symmetry, however, makes explicit the essential feature of a coherent superposition in contradistinction to a mixture: the superposition is not more mixed than the constituent states. In order to extend the notion of a coherent superposition also to nonpure states, we have, therefore, to use (42) instead of (39). That for nonpure states (39) is no longer equivalent to (42) but even almost meaningless is demonstrated by taking a faithful state for ρ_1: then no ρ_3 satisfies (42) whereas (39) imposes no restriction at all to ρ_3.

3.2. Definition. Let ρ_i be states of a W^* algebra with supports S_i, $i = 1, 2, 3$. We write

$$K(\rho_1, \rho_2, \rho_3), \text{ if } S_1 \wedge S_2 = S_1 \wedge S_3 = S_2 \wedge S_3 = 0 \text{ and}$$
$$S_1 \vee S_2 = S_1 \vee S_3 = S_2 \vee S_3 \tag{43}$$

and say then that the three states satisfy the coherence relation or that they constitute a coherent triple. We also write

$$K(\rho_1, \rho_2), \text{ if there is a state } \rho_3 \text{ with } K(\rho_1, \rho_2, \rho_3) \tag{44}$$

and say then, that ρ_1 and ρ_2 are coherently superposable.

By means of (43) and (44) we have introduced a symmetric ternary and a symmetric binary relation which are both denoted by the symbol K but discriminated by the number of components in the argument.

In passing we note that the second part of (43) is equivalent to the set of implications

$$\rho_i(P) = \rho_j(P) = 1 \ (=0) \text{ implies } \rho_k(P) = 1 \ (=0) \tag{45}$$

where P is an element in $P(\mathfrak{M})$ and (i, j, k) runs through all the permutations of $(1, 2, 3)$. This again illustrates how much stronger the condition (43) is than (39) for nonpure states.

The decisive structural classification of the coherence relation is obtained from the observation that (43) is valid, iff the supports are pairwise strongly perspective via the third one from the application of Fillmore's results.

3.3. Theorem. (i) Let ρ_i be states of a W^* algebra with supports S_i, $i=1,2,3$. Then

$$K(\rho_1,\rho_2), \text{ iff } S_1\wedge S_2=0 \quad\text{and}\quad S_1\sim S_2 \tag{46}$$

which implies $S_1 \overset{u}{\sim} S_2$. If $K(\rho_1,\rho_2,\rho_3)$, then the supports are pairwise unitarily equivalent and the central supports are equal.

(ii) If ρ is a state of a W^* algebra, then there exists a state φ such that $K(\rho,\varphi)$, iff $S_\rho \prec S_\rho^\perp$.

(iii) If P is a projection of W^* algebra, then there exists a projection Q such that $C(P,Q)=0$, iff $P\sim P^\perp$. It follows that $P\overset{u}{\sim}P^\perp\overset{u}{\sim}Q^\perp\overset{u}{\sim}Q$.

Proof. (i) $K(\rho_1,\rho_2)$ implies the strong perspectivity of S_1 and S_2, thus the unitary equivalence of S_1 and S_2 by Fillmore's theorem, and hence the equivalence of S_1 and S_2. On the other hand, the right-hand side of (46) gives the strong perspectivity of S_1 and S_2 by Fillmore's lemma, say via a projection S_3. Since $S_3\leqslant S_1\vee S_2$ and $S_1\vee S_2$ is σ-finite [being the support of $\lambda\rho_1+(1-\lambda)\rho_2$ for any $0<\lambda<1$], S_3 is σ-finite and there is a (normal) state ρ_3 supported by S_3. This shows that $K(\rho_1,\rho_2)$ holds. It is now clear that $K(\rho_1,\rho_2,\rho_3)$ implies the unitary equivalence of the supports, and this implies immediately that the central supports are equal.

(ii) Assume $S_\rho\prec S_\rho^\perp$; then there is a projection P such that $S_\rho\sim P\leqslant S_\rho^\perp$. It follows that $S_\rho\wedge P=S_\rho P=0$, and the right-hand side of (46) is satisfied, with $P=S_2$ which is σ-finite. Conversely, assume $K(\rho,\varphi)$. Then S_ρ and S_φ are strongly perspective via R, so $R\overset{u}{\sim}S_\rho$. Applying (22) we have $R=1-R^\perp$ $=(S_\rho\wedge R)^\perp-R^\perp=(S_\rho^\perp\vee R^\perp)-R^\perp\sim S_\rho^\perp-(S_\rho^\perp\wedge R^\perp)$, and then $S_\rho\sim$ $S_\rho^\perp-(S_\rho^\perp\wedge R^\perp)\leqslant S_\rho^\perp$.

(iii) Suppose $P\sim P^\perp$, then by Fillmore's Lemma P and $P^\perp$ are strongly perspective via a Q, that is to say $P\wedge Q=P^\perp\wedge Q=0$ and $P\vee Q=P^\perp\vee Q=P^\perp\vee P=1$. Taking orthocomplements, $P^\perp\wedge Q^\perp=P$ $\wedge Q^\perp=0$, and $C(P,Q)=0$. On the other hand, if $C(P,Q)=0$, then P and $P^\perp$ are strongly perspective via Q, hence perspective, hence unitarily equivalent. Furthermore, P and Q are strongly perspective via $P^\perp$ so again $P\overset{u}{\sim}Q$. ■

3.4. Example. Consider a traditional quantum mechanical description with $\mathfrak{M}\subseteq\mathfrak{B}(\mathfrak{H})$ such that $\mathfrak{M}'(\subset\mathfrak{M})$ is a commutative totally atomic von Neumann algebra. For three given unit vectors $\Psi_i\in\mathfrak{H}$ define $\Psi_i^{(n)}=P_n\Psi_i$, $i=1,2,3$, where the P_n's are the atoms of $\mathfrak{M}'$ (i.e., of the center of $\mathfrak{M}$). In order that the states ρ_i on $\mathfrak{M}$ defined by the state vectors Ψ_i satisfy the coherence relation (43) the central supports $C_i=\Sigma_n P_n$, all n with $\Psi_i^{(n)}\neq 0$, must necessarily be equal. This is the case, iff the components $\Psi_i^{(n)}$ are nonvanishing for the same indices say $n=m$. Relation (44) is then fulfilled, iff

$$\Psi_1^{(m)} \neq \Psi_2^{(m)} \neq \Psi_3^{(m)} \quad \text{and} \quad \Psi_3^{(m)} = c_1^{(m)}\Psi_1^{(m)} + c_2^{(m)}\Psi_2^{(m)} \tag{47}$$

for all m, where $c_i^{(m)} \in \mathbb{C}$. [(47) is a special case of Theorem 3.8].

Whereas the negative statement, namely, ρ_1 and ρ_2 are not coherently superposable if Ψ_1 and Ψ_2 are in different coherent subspaces $P_n\mathfrak{H}$, is commonly accepted, the positive interpretation of (47) as a form of a coherence relation seems not to be in use at all. For an illustration of (47) imagine an experimental device which produces a beam in which two different kinds of particles occur in a fixed ratio and all particles of the same kind are in one and the same pure state. After separation it is possible to perform the double slit experiment with each subbeam and to observe interference phenomena. Thus we have a realization of (47) where every vector has two components, one for each kind of particle, and Ψ_i, $i = 1, 2$, are the states where the slits with the number i are open, and Ψ_3 is the state where both slits for each subbeam are open. It is of course only a matter of taste to call this experimental arrangement a coherent superposition of states of the total system. But already in this case where one relies on an explicit Hilbert space description the proposed coherence relation provides a concise formulation of rather involved, experimentally meaningful relationships.

The immediate idea, that our coherence relation, in which states are superposed without increasing the degree of mixedness, is only possible with incompatible observables, is made precise as follows.

3.5. Proposition. (i) If for a given triple of states of a W^* algebra one of the supports commutes with the other two, then the coherence relation does not hold. (ii) Given two states ρ_1 and ρ_2 of a W^* algebra with supports S_1 and S_2 such that $S_1 \wedge S_2 = 0$ and for every pair of states $\bar{\rho}_1$ and $\bar{\rho}_2$ with supports $\bar{S}_1 \leq S_1$ and $\bar{S}_2 \leq S_2$ $K(\bar{\rho}_1, \bar{\rho}_2)$ is false, it follows that $S_1 S_2 = 0$.

Proof. (i) Assume $K(\rho_1, \rho_2, \rho_3)$ and $[S_1, S_2]_- = [S_1, S_3]_- = 0$. Then by (43) we derive $S_1 = (S_1 \vee S_2) \wedge S_1 = (S_3 \vee S_2) \wedge S_1 = (S_3 \wedge S_1) \vee (S_2 \wedge S_1) = 0$ by distributing as we may (cf. Holland, 1970; p. 80). But being a support $S_1 \neq 0$. (ii) The assumption amounts to saying that if $\bar{S}_i \leq S_i$ for $i = 1, 2$, then $\bar{S}_1 \nsim \bar{S}_2$ because $\bar{S}_1 \wedge \bar{S}_2 = 0$ follows. Using (22)

$$\bar{S}_1 = S_1 - (S_1 \wedge S_2^\perp) \sim (S_1 \vee S_2^\perp) - S_2^\perp = S_2 - (S_1^\perp \wedge S_2) = \bar{S}_2$$

we have a contradiction unless $\bar{S}_1 = \bar{S}_2 = 0$, which means that $S_1 \leq S_2^\perp$, or equivalently $S_1 S_2 = 0$. ∎

It is interesting to note that the intimate relation between incompatibility and coherence exemplified by Proposition 3.5 is due to the orthomodular lattice structure of $P(\mathfrak{M})$ alone. But the lattice $P(\mathfrak{M})$ determines a large part of the algebraic structure of $\mathfrak{M}$.

3.6. Proposition. Let $\mathfrak{M}$ be a W^* algebra. Then $\mathfrak{M}$ is Abelian, iff $K=\varnothing$.

Proof. Observe first that the ternary K relation is void, iff the binary K relation is void. Now, if $\mathfrak{M}$ is Abelian, then all support projections commute, and by Proposition 3.5(i) the ternary K relation can never hold. Assume, conversely that the binary K relation is empty. Then by Proposition 3.5(ii) $[P,Q]_-=0$ for all σ-finite projections P and Q with $P\wedge Q=0$. If P and Q are arbitrary σ-finite projections, then $P'=P-(P\wedge Q)$ and $Q'=Q-(P\wedge Q)$ have the same commutator as P and Q, are σ-finite, and have trivial intersection. Thus, all σ-finite projections of $\mathfrak{M}$ commute. If P is not σ-finite, there exists an increasing net $\{P_\gamma\}$ of σ-finite projections P_γ with $P=\vee P_\gamma$, and by the $\sigma(\mathfrak{M},\mathfrak{M}_*)$-continuity of the left- and right-multiplication map, P commutes with every σ-finite projection. Repeating the argument, we conclude that all projections commute, which implies that $\mathfrak{M}$ is Abelian. ■

In order to show that states with maximal incompatible supports have strong coherence properties, let us denote by $\rho^\perp$ a state which has support $S_\rho^\perp$ for a given state ρ with support $S_\rho\neq 1$ in a σ-finite W^* algebra (the definition makes no sense otherwise).

3.7. Proposition. For two given states ρ_1 and ρ_2 of a σ-finite W^* algebra with supports S_1 and S_2, the following conditions are equivalent:

(i) $K(\rho_1,\rho_2,\rho_1^\perp)$

(ii) $C(S_1,S_2)=0$ [cf. (18)]

These conditions imply (iii): There is no nontrivial ($\neq 0$ and 1) projection which is truth-definite for ρ_1 and ρ_2.

Proof. Assume (i); then $S_1\wedge S_2=S_2\wedge S_1^\perp=0$ and $S_1\vee S_2=S_2\vee S_1^\perp=S_1\vee S_1^\perp=1$, and orthocomplementation gives $S_1\wedge S_2^\perp=S_1^\perp\wedge S_2^\perp=0$, whence $C(S_1,S_2)=0$, which is (ii). Assume (ii); then using a shorthand notation for the four possibilities involved, $S_1^{(\perp)}\wedge S_2^{(\perp)}=0$ and $S_1^{(\perp)}\vee S_2^{(\perp)}=1$, which shows that the triple $(S_1,S_2,S_1^\perp)$ satisfies (43). Since by assumption on $\mathfrak{M}$, $S_1^\perp$ supports some state, (i) follows. Assume (ii), then if

P is truth-definite in ρ_1 and ρ_2, we have four cases:

$$\begin{aligned}
&(\mathrm{TT})\ S_1 \leqslant P \quad \text{and} \quad S_2 \leqslant P \Rightarrow 1 = S_1 \vee S_2 \leqslant P \Rightarrow P = 1\\
&(\mathrm{TF})\ S_1 \leqslant P \quad \text{and} \quad P \leqslant S_2^{\perp} \Rightarrow S_1 \leqslant S_2^{\perp} \Rightarrow S_1 = S_1 \wedge S_2^{\perp} = 0\\
&(\mathrm{FT})\ P \leqslant S_1^{\perp} \quad \text{and} \quad S_2 \leqslant P \Rightarrow S_2 \leqslant S_1^{\perp} \Rightarrow S_2 = S_2 \wedge S_1^{\perp} = 0\\
&(\mathrm{FF})\ P \leqslant S_1^{\perp} \quad \text{and} \quad P \leqslant S_2^{\perp} \Rightarrow P \leqslant S_1^{\perp} \wedge S_2^{\perp} = 0 \Rightarrow P = 0
\end{aligned}$$

Since (TF) and (FT) contradict the assumption that S_1 and S_2, respectively, are supports, we obtain (iii). ■

A simple example, $S_1 = \mathrm{diag}(1,1,0)$ and $S_2 = \mathrm{diag}(0,1,1)$ in the (3×3) matrices, shows that (iii) does not imply (ii) in Proposition 3.7.

Both for technical as well as interpretational purposes it is useful to reduce the coherence relation for arbitrary states to that of factor states.

3.8. Proposition. Let $\mathfrak{M}$ be a W^* algebra with a separable predual, μ the central measure corresponding to some faithful (normal) state of $\mathfrak{M}$ (cf. Section 2) by means of which the states and observables are decomposed, and $\rho_i = \int^{\oplus} \rho_i^{\omega}\, d\mu\,(\omega)$, $i = 1,2,3$, be states of $\mathfrak{M}$. Then

$$K(\rho_1, \rho_2, \rho_3), \qquad \text{iff } K(\rho_1^{\omega}, \rho_2^{\omega}, \rho_3^{\omega})\,\mu\text{-a.e.} \tag{48}$$

$$K(\rho_1, \rho_2), \qquad \text{iff } K(\rho_1^{\omega}, \rho_2^{\omega})\,\mu\text{-a.e.} \tag{49}$$

Proof. Follows directly from Proposition 2.7. ■

Note that (48) is just the generalization of (47) to nonpure states, and W^* algebras with nonatomic centers.

Another notion which has to be generalized from traditional quantum mechanics to our algebraic formalism is that of a coherent subspace. For pure states a coherent sector corresponding to a given pure state ρ should consist of all states which are coherently superposable with ρ (they are automatically pure) together with ρ itself. Since for different pure states ρ and φ one has automatically $S_\rho \wedge S_\varphi = 0$, the characterization of the coherent sector $\mathcal{C}(\rho)$ of a pure state can be given directly in terms of the coherence relation as follows:

$$\begin{aligned}
\mathcal{C}(\rho) &= \{\varphi \in \mathfrak{S}_n : K(\rho, \varphi)\} \cup \{\rho\}\\
&= \{\varphi \in \mathfrak{S}_n : S_\varphi \overset{u}{\sim} S_\rho \quad \text{and} \quad S_\varphi \wedge S_\rho = 0\} \cup \{\rho\}\\
&= \{\varphi \in \mathfrak{S}_n : S_\varphi \overset{u}{\sim} S_\rho\}\\
&= \{\varphi \in \mathfrak{S}_n : \text{there exists a unitary element } U \in \mathfrak{M}\\
&\qquad \text{such that } \varphi(A) = \rho(U^* A U) \text{ for all } A \in \mathfrak{M}\}.
\end{aligned} \tag{50}$$

Although the first two equalities in (50) are always true, one cannot conclude from $S_\varphi = U^* S_\rho U$ that either $S_\varphi = S_\rho$ or $S_\varphi \wedge S_\rho = 0$ if S_ρ is not an atom (i.e., ρ is not pure). The different forms of (50) are no longer equivalent for nonpure states. In our opinion it is natural to say that a state φ belongs to $\mathcal{C}(\rho)$ if there is a finite chain beginning at ρ, ending at φ, and such that adjacent states of the chain are in the binary K relation. That is ρ and φ are involved in an interference phenomenon. More precisely, we have the following.

3.9. Definition. Two states ρ and φ of a W^* algebra are said to be in the interference relation $I(\rho, \varphi)$ if either $K(\rho, \varphi)$ or there exists a finite set of n states $\{\chi_1, \chi_2, \ldots, \chi_n\}$ such that $K(\rho, \chi_1)$, $K(\chi_1, \chi_2), \ldots, K(\chi_n, \varphi)$ is valid. The latter relation will be denoted by $K^{n+1}(\rho, \varphi)$.

As a product of symmetric binary relations $I(.,.)$ is symmetric. $I(.,.)$ is of course transitive but not necessarily reflexive, i.e., $I(\rho, \rho)$ may be valid or not. We can now define the coherent sector corresponding to a state as follows:

3.10. Definition. The coherence class of a state ρ of a W^* algebra is $\mathcal{C}(\rho) = \{\varphi \in \mathfrak{S}_n : I(\rho, \varphi)\}$.

It is easily verified that for pure states (50) coincides with Definition 3.10. The definition of I guarantees that the state space $\mathfrak{S}_n$ can be partially partitioned into well-defined, nonintersecting coherence classes. $\mathcal{C}(\rho)$ is empty iff ρ does not belong to $\mathcal{C}(\rho)$, iff S_ρ has nontrivial intersection with every unitary equivalent projection. As a straightforward consequence of the definitions and (7) we obtain the following proposition.

3.11. Proposition. Let ν be a structural symmetry of a W^*-algebraic description (i.e., ν is an affine bijection of the state space, or a Jordan-$*$-automorphism). Then ν maps coherent triples into coherent triples and coherence classes into coherence classes.

4. COHERENCE IN FACTORS

According to Proposition 3.8 the coherence relations in a decomposable W^*-algebra can be reduced to those of factors. We will now analyze the coherence relations in σ-finite factors. Throughout this section, $\mathfrak{F}$ is a σ-finite factor. In order to classify the relevant unitary equivalence classes in $P(\mathfrak{F})$ we first identify the classes of equivalent projections $\mathfrak{E}_d$ by means of the value d of the dimension function. Only in the case that $d = \infty$ does unitary equivalence constitute an additional condition by stipulating that also the complementary projections have constant dimension, because only in this case is $c = D(P^{\perp}) = D(1) - D(P)$ not fixed by the value of $D(P)$. Thus we will only in this case make explicit the value c of this "codimension" in the notation $\mathfrak{A}_d^{(c)}$ of a unitary equivalence class of projections.

By Theorem 3.3(ii), a necessary and sufficient condition that one (and hence every) element in a unitary equivalence class $\mathfrak{A}_d^{(c)}$ of $P(\mathfrak{F})$ have a nonintersecting partner is

$$d \leqslant D(1)/2, \qquad \text{if } \mathfrak{F} \text{ is finite} \tag{51}$$

$$c = \infty, \qquad \text{if } \mathfrak{F} \text{ is infinite} \tag{52}$$

A rather unexpected result of our discussions is the following theorem, which shows that type III factors have strong coherence properties. They are almost sufficient to characterize these factors.

4.1. Theorem. Let $\mathfrak{M}$ be a σ-finite W^* algebra. The following conditions are equivalent:

(i) $\mathfrak{M}$ is a factor of type I_2 or III.
(ii) For each projection P of $\mathfrak{M}$ with $0 < P < 1$ there exists a projection Q such that $C(P, Q) = 0$.
(iii) For each nonfaithful state ρ of $\mathfrak{M}$ there exist states φ and χ of $\mathfrak{M}$ such that $K(\rho, \varphi, \chi)$.
(iv) Every pair of states ρ and φ of $\mathfrak{M}$ with $S_\rho \wedge S_\varphi = 0$ satisfies $K(\rho, \varphi)$.

Proof. (α) Every projection P of the I_2 factor with $0 < P < 1$ is an atom. Thus every state is either pure or faithful. To deduce (ii) take any atom Q different from P and $P^{\perp}$; then $P, P^{\perp}, Q, Q^{\perp}$ are all different atoms of I_2. $C(P, Q) = 0$ follows. To obtain (iii), choose any two pure states φ and χ such that ρ, φ, and χ are mutually different. (iv) is established analogously.

(β) If $\mathfrak{M}$ is a factor of type III, we use the fact (Sakai, 1971; Proposition 2.2.14) that all nonzero projections in a σ-finite factor of type III are equivalent and thus all projections P with $0 < P < 1$ are unitarily equivalent (because their complements are equivalent). Theorem 3.3(iii) gives (ii). If ρ is not-faithful, then $S_\rho \overset{u}{\sim} S_\rho^{\perp}$. Taking φ such that $C(S_\rho, S_\varphi) = 0$, we have $K(\rho, \rho^{\perp}, \varphi)$. This produces (iii). Again, $S_\rho \wedge S_\varphi = 0$ gives nontriviality of both S_φ and S_ρ and then $S_\varphi \overset{u}{\sim} S_\rho$. Theorem 3.3(i) gives (iv).

(α) and (β) show that (i) implies the rest.

(γ) Assume conversely that $\mathfrak{M}$ is not a factor of type I_2 or III. Then either $\mathfrak{M}$ is not a factor, or it is a factor of one of the following types: I_n ($n \geqslant 3$), II. If $\mathfrak{M}$ is not a factor, we have a central projection P with $0 < P < 1$. $C(P, Q) = 1$ for every projection Q by (18), and a state supported by P cannot belong to a coherent triple by Proposition 3.5(i). The factor cases can be excluded by use of Theorem 3.3(ii) and (iii). ■

The property which is shared by states lying in the same coherence class of $\mathfrak{F}$ is made explicit in the following result:

4.2. Theorem. Given two states ρ and φ of $\mathfrak{F}$, the following conditions are equivalent:

(i) $I(\rho, \varphi)$

(ii) $K^2(\rho, \varphi)$

(iii) $S_\rho \overset{u}{\sim} S_\varphi$ and $K(\rho, \chi)$ for some state χ of $\mathfrak{F}$.

In particular $K^2(.,.)$ is transitive.

Proof. The implications (ii)$\Rightarrow$(i)$\Rightarrow$(iii) are clear for any W^* algebra. Assume (iii). To prove $K^2(\rho, \varphi)$ we have, in view of Theorem 3.3(i), to establish the existence of a projection P of $\mathfrak{F}$ such that $P \sim S_\rho$ and $P \wedge S_\rho = P \wedge S_\varphi = 0$. Since $K(\rho, \chi)$ holds, Theorem 3.3(ii) implies $S_\rho \prec S_\rho^\perp$ and hence by unitary equivalence $S_\varphi \prec S_\varphi^\perp$. We have two cases:

(α) $S_\rho \vee S_\varphi$ is finite: We first prove that in this case $S_\rho \wedge S_\varphi \prec S_\rho^\perp \wedge S_\varphi^\perp$. If $S_\rho^\perp$ (hence $S_\varphi^\perp$) is infinite, then $\mathfrak{F}$ must be infinite, and then $S_\rho^\perp \wedge S_\varphi^\perp = (S_\rho \vee S_\varphi)^\perp$ is infinite as complement of a finite projection in an infinite factor. Whence $S_\rho \wedge S_\varphi \prec S_\rho^\perp \vee S_\varphi^\perp$ by finiteness of $S_\rho \wedge S_\varphi$. If $S_\rho^\perp$ is finite, then $\mathfrak{F}$ is finite, and application of (27) gives

$$\begin{aligned} D(S_\rho \wedge S_\varphi) &= D(S_\rho) + D(S_\varphi) - D(S_\rho \vee S_\varphi) \\ &\leqslant D(S_\rho^\perp) + D(S_\varphi) - D(S_\rho \vee S_\varphi) \\ &= D(S_\rho^\perp) + D(S_\rho) - D(S_\rho \vee S_\varphi) \\ &= D(1) - D(S_\rho \vee S_\varphi) \\ &= D\big((S_\rho \vee S_\varphi)^\perp\big) \end{aligned}$$

using $D(S_\varphi) = D(S_\rho) \leqslant D(S_\rho^\perp)$. We thus have a Q such that $(S_\rho \wedge S_\varphi) \sim Q \leqslant S_\rho^\perp \wedge S_\varphi^\perp$. Now by Sakai (1971; Proposition 2.4.2), $A = S_\rho - (S_\rho \wedge S_\varphi) \sim S_\varphi - (S_\rho \wedge S_\varphi) = B$, and by Fillmore's Lemma A and B are strongly perspective, say via R, since $A \wedge B = 0$. Since $R \leqslant R \vee A = R \vee B = A \vee B \leqslant S_\rho \vee S_\varphi$, $R \perp (S_\rho \vee S_\varphi)^\perp$, and then $R \perp Q$. Let $P = R \oplus Q$ (the symbol $\oplus$ stands for $\vee$ when connecting orthogonal elements). Since $S_\rho \wedge S_\varphi \perp A$ (respectively, B) we may add the equivalences $A \sim R \sim B$ and $S_\rho \wedge S_\varphi \sim Q \sim S_\rho \wedge S_\varphi$, to obtain $S_\rho = A \oplus (S_\rho \wedge S_\varphi) \sim R \oplus Q = P \sim B \oplus (S_\rho \wedge S_\varphi) = S_\varphi$. Finally, it is easily shown that $(B \vee R) \perp ((S_\rho \wedge S_\varphi) \vee Q)$ and that $(A \vee R) \perp ((S_\rho \wedge S_\varphi) \vee Q)$, so that Holland (1964; Lemma 7) gives $S_\rho \wedge P = (A \oplus (S_\rho \wedge S_\varphi)) \wedge (R \oplus Q) = (A \wedge R) \oplus (S_\rho \wedge S_\varphi \wedge Q) = 0$ and (analogously) $S_\varphi \wedge P = 0$.

(β) $S_\rho \vee S_\varphi$ is infinite: We have S_ρ (hence S_φ) are infinite, for otherwise Sakai (1971; Corollary 2.5.5) would lead to $S_\rho \vee S_\varphi$ is finite. Since $S_\rho \prec S_\rho^\perp$, $S_\rho^\perp$ is infinite and then $S_\rho \sim S_\rho^\perp$. By Fillmore's Theorem S_ρ and S_φ are perspective via a P, i.e., $P \wedge S_\rho = P \wedge S_\varphi = 0$ and $P \vee S_\rho = P \vee S_\varphi = 1$. Equation (22) gives $P = P - (P \wedge S_\rho) \sim (P \vee S_\rho) - S_\rho = 1 - S_\rho = S_\rho^\perp \sim S_\rho$. ■

4.3. Corollary. Given a state ρ of $\mathfrak{F}$, then either $K(\rho, \chi)$ for some state χ of $\mathfrak{F}$ and then $\mathcal{C}(\rho) = \{\varphi \in \mathfrak{S}_n(\mathfrak{F}) : S_\varphi \overset{u}{\sim} S_\rho\}$, or $\mathcal{C}(\rho) = \varnothing$.

We have thus recovered partially (50) which is valid for pure states. However, one cannot conclude from $\varphi \in \mathcal{C}(\rho)$ that $\varphi(A) = \rho(U^* A U)$ for some unitary elements $U \in \mathfrak{F}$ and all $A \in \mathfrak{F}$, as one easily sees in the I_4 factor for nonpure states.

It would be interesting to know if the K^2 relation is transitive in the general case of W^* algebras or even orthomodular lattices, that is, if there is a purely lattice theoretic proof of Theorem 4.2 without the assumption of a trivial center.

The unitary equivalence classes $\mathfrak{U}_d^{(c)}$ specify the coherence classes via

$$\mathcal{C}_d^{(c)} = \{\varphi \in \mathfrak{S}_n(\mathfrak{F}) : S_\varphi \in \mathfrak{U}_d^{(c)}\}, \qquad 0 < d \leq c \tag{54}$$

in σ-finite factors.

In Table I we have listed all coherence classes in the various factors, showing for comparison the equivalence and unitary equivalence classes of projections which derive immediately from the possible values of the dimension and codimension. In the last column we have included the usual coherence class of pure states occurring only in type-I factors. The compari-

TABLE I

Type of factor	Equivalence classes	Unitary equivalence classes	Coherent sectors	Pure st. sectors
I_2	$\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2$	$\mathfrak{U}_0, \mathfrak{U}_1, \mathfrak{U}_2$	$\mathcal{C}_1$	$\mathcal{C}_1$
I_3	$\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2, \mathfrak{E}_3$	$\mathfrak{U}_0, \mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3$	$\mathcal{C}_1$	$\mathcal{C}_1$
I_n	$\mathfrak{E}_d, 0 \leq d \leq n$	$\mathfrak{U}_d = \mathfrak{E}_d, 0 \leq d \leq n$	$\mathcal{C}_d, 1 \leq d \leq \frac{1}{2}n$	$\mathcal{C}_1$
I_∞	$\mathfrak{E}_d, 0 \leq d \leq \infty$	$\mathfrak{U}_d = \mathfrak{E}_d, 0 \leq d < \infty$	$\mathcal{C}_d, 1 \leq d < \infty$	
		$\mathfrak{E}_\infty \supset \mathfrak{U}_\infty^c, 0 \leq c \leq \infty$	$\mathcal{C}_\infty^\infty$	$\mathcal{C}_1$
II_1	$\mathfrak{E}_d, d \in [0, 1]$	$\mathfrak{U}_d = \mathfrak{E}_d, d \in [0, 1]$	$\mathcal{C}_d, d \in (0, \frac{1}{2}]$	none
II_∞	$\mathfrak{E}_d, d \in [0, \infty]$	$\mathfrak{U}_d = \mathfrak{E}_d, d \in [0, \infty)$	$\mathcal{C}_d, d \in (0, \infty)$	none
		$\mathfrak{E}_\infty \supset \mathfrak{U}_\infty^c, c \in [0, \infty]$	$\mathcal{C}_\infty^\infty$	
III	$\mathfrak{E}_0, \mathfrak{E}_\infty$	$\mathfrak{U}_0, \mathfrak{U}_\infty^\infty, \mathfrak{U}_\infty^0$	$\mathcal{C}_\infty^\infty$	none

TABLE II

Type of factor	Nonfaithful states which are not coherently superposable?	Nonpure states which are coherently superposable?	Are there ρ, φ with $S_\rho \wedge S_\varphi = 0$ which are not coherently superposable?	Necessary and sufficient condition on S_ρ for coherent superposition
I_2	No	No	No	$S_\rho \neq 1$
I_3	No	No	Yes	$D(S_\rho) = 1$
$I_{n \geq 4}$	Yes, those with $D(S_\rho) > \frac{1}{2}n$	Yes	Yes	$0 < D(S_\rho) \leq \frac{1}{2}n$
I_∞	Yes, those with $D(S_\rho^\perp) < \infty$	Yes	Yes	$D(S_\rho^\perp) = \infty$
II_1	Yes, those with $D(S_\rho) > 1/2$	Yes, all coherent triples are nonpure	Yes	$D(S_\rho) \leq 1/2$
II_∞	Yes, those with $D(S_\rho) < \infty$	Yes, all coherent triples are nonpure	Yes	$D(S_\rho^\perp) = \infty$
III	No	Yes, all coherent triples are nonpure	No	$S_\rho \neq 1$

son of columns 4 and 5 shows the increase of complexity which one encounters in dealing with coherence relations also for nonpure states. Table II contains in a similar pattern the answer to some formal questions concerning the coherence relations in factors.

5. CONCLUSIONS

Having elaborated the formal properties of this form of the coherence relation, which was obtained from the pure state superposition relation in a symmetric form, we may attempt a global analysis of the physical meaning. (A detailed analysis would depend on specific states of specific models.) By means of purely lattice theoretical reasoning we found that coherent superposability (in the sense of Definition 3.2) and (principal) incompatibility of the support projections are almost equivalent conditions on the given states: In a coherent triple every support must be incompatible at least with one of the other two supports, and if the supports of two states are incompatible then there are two substates (the supports of which are respectively smaller than those of the given states) which are coherently superposable (cf. Proposition 3.5). But in a W^* algebra there is a pair of incompatible observables only if there is a pair of incompatible support projections. In order to show the principal incompatibility of two observables one has to discuss all known and even all conceivable experimental devices for the measurement of these quantities (with the result that no two of them allow for the precise and simultaneous determination of their values). The touch of Gedankenexperiment and extrapolation is typical for this kind of argument. In contradistinction to this, the coherent superposition of two states requires for confirmation only one positive experiment with interference phenomena. Thus, in spite of their similar formal status, coherent superposition seems to be a much more easily accessible effect than incompatibility and deserves to be placed in the center of a foundational investigation of quantum physics. For this reason it is also gratifying that the concept of maximal incompatibility can be connected with coherence effects (Proposition 3.7).

It is interesting to characterize those pure or nonpure states which are in a direct or indirect coherence relation to each other and constitute thus a generalization of the notion of a coherent sector. In Theorem 4.2, it is shown that all direct or indirect coherence relations reduce to $K^2(\rho, \varphi)$ and that this corresponds to unitary equivalence of the supports provided that there is at least one unitary equivalent projection with zero intersection with one of the supports. One may consider this as the essential part of the property of being coherently superposable, and this part is thus shown to be a transitive relation. For this result special features of the projection lattice $P(\mathfrak{M})$ came into play which are not available in general orthomodular lattices.

A central point, not only for technical reasons, is the decomposability of the coherence relation into a set of relations for factor states. Let us emphasize that it is not artificial to formulate the coherence relation also for nonfactorial states which do not have dispersion-free values of the classical observables. In traditional nonrelativistic quantum mechanics one may argue that this is merely a shorthand notation for describing some complicated experimental situations (Example 3.4). But in elementary particle physics and in many-body physics there are completely natural state preparations, which do not fix the values of the classical observables. If the central decomposition of the involved states leads to pure states, then the coherent superposition of the classically mixed states is in fact reduced to the usual coherent superposition of pure states and has to be interpreted correspondingly. If the factor states in the decomposition are mixed states of a type-I W^* algebra then they can be written as a countable convex sum of pure states of this algebra. In contrast to the factor decomposition (by means of a central measure) this further decomposition into pure states is never unique and the usual interpretational difficulties of mixed states in traditional quantum mechanics render also the discussion of coherence more intricate; cf., e.g., Ludwig (1964). As a general feature one may state that the coherence relation for the mixed states expresses properties which are common to all ensembles of pure states which may show up in the spectral decompositions of the involved density operators. As shown by (10), the supports of the mixed states ρ_i are the projections on the subspaces V_i spanned by the eigenvectors of the density matrices corresponding to the ρ_i in any Hilbert space representation. With this in mind, we can express the coherence relation $K(\rho_1, \rho_2, \rho_3)$ in the way that every pure state in a decomposition of ρ_3 (and given by $\Psi_3 \in V_3$) is a coherent superposition of pure states in certain decompositions of ρ_1 and ρ_2 (which are given by vectors $\Psi_1 \in V_1$ and $\Psi_2 \in V_2$), but no vector in V_i is a linear combination of vectors in V_j or V_k alone, where (i, j, k) is a permutation of $(1,2,3)$. We see that in this case it is a complicated matter to express the content of $K(\rho_1, \rho_2, \rho_3)$ by pure state coherence relations. In our opinion, however, the coherence relation for the considered mixed states has a value of its own: it decouples the experimentally observable coherence phenomena from the pure state idealization. This should be viewed necessary for the consistency of the theoretical formalism. It is impossible to prepare experimentally a pure state with complete precision and to discriminate it from neighboring mixed states. Thus, observable effects such as interference phenomena should not depend too sensitively on the purity of the states involved. That coherent light is not dependent on a monochromatic pure state preparation is well known in quantum optics (Nussenzweig, 1973).

If the central decomposition of the states in a coherent triple leads to factor states of W^* algebras of type II or III, a further reduction to pure states is not possible at all. Since type II factors occur only in the infinite temperature representations, the physically interesting case are the type III factors which abound in many body physics via thermodynamic limit. A look at Table I or Table II shows that in these factors all states with nonintersecting supports are coherently superposable and that, correspondingly, there are many maximally incompatible yes–no observables (cf. Theorem 4.1). This is perhaps the most important and surprising result of our investigation and may lead to new insights into the quantum mechanics of many-body systems. It deserves of course a thorough elaboration and has to be tested in concrete models. Let us only mention here that apparently the limit to infinitely many degrees of freedom makes obsolete many necessary conditions which restrict the possibility of coherent superposition for finite quantum systems.

If as usual the dynamics, in the Schrödinger picture, is given by a one-parameter group of structural symmetries, then two coherently superposable states retain this property for all (finite) times. Coherence may be broken only by a generalized kind of dynamics or in the asymptotic limit $t \to \infty$. Simple dynamical models for each kind of breaking coherence, and thus making measurement objective are indeed available (Primas, 1971; Hepp, 1972).

Let us end our discussion with a remark on the superposition principle, which has not been mentioned throughout our study of the coherence relation because we always assumed that the W^*-algebraic description (i.e., the W^* algebra) had been given at the outset. How many possibilities of coherent state superpositions can be realized is then expressed by the family of coherence classes in the given algebra and depends, e.g., on the number of classical observables. A superposition principle is only required if one tries to reconstruct the algebraic structure from experimental data, and for this purpose the analysis of the possible coherent interference phenomena may play indeed a decisive role because they give information on how many observables are incompatible or classical.

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